FPR = **F**alse **P**roduct **R**ule

We all dutifully learn the product rule the correct way:

$$\frac{d}{dx}[f(x)g(x)] \;=\; f'(x)g(x) + f(x)g'(x).$$

Still, who doesn't participate in a little lip service to rules we break in practice, right? Well no matter the temptation to do otherwise, align your practice with theory when it comes to differentiating a product. I know it is awfully tempting to write

$$\frac{d}{dx}\left[(x^4 - 3x^2)e^{4x-5}\right] \underset{\text{False!}}{=} (4x^3 - 6x)(4)e^{4x-5} \qquad \text{FPR},$$

as if

$$\frac{d}{dx}\left[f(x)g(x)\right] \underset{\text{False!}}{=} f'(x)g'(x) \qquad \text{FPR}$$

were, indeed, true, but don't do it! You will be caught, and it will go on your record! Don't even do it when the correct use of the product rule would make your expression so much longer, as in

$$\frac{d}{dx}\left[(x^2\cos x - e^x \sin x)\ln(x^2+1)\right] \underset{\text{False!}}{=} (-2x\sin x - e^x\cos x)\cdot\frac{2x}{x^2+1}. \qquad \text{FPR}$$

RFPRF = **R**everse **F**alse **P**roduct **R**ule is **F**alse

So, you spent all that mental energy learning the product rule and how to apply it in practice. Don't lose your grip on it when you need to antidifferentiate! If it ain't so that

$$\frac{d}{dx}[e^{-2x}\sin(3x)] \underset{\text{False!}}{=} -2e^{-2x}\cdot 3\cos(3x), \qquad \text{FPR}$$

then it also is wrong to say

$$\int -2e^{-2x}\cdot 3\cos(3x)\,dx \underset{\text{False!}}{=} e^{-2x}\sin(3x) + C. \qquad \text{RFPRF}$$

Speaking more generally, avoid

$$\int f'(x)g'(x)\,dx \underset{\text{False!}}{=} \left(\int f'(x)\,dx\right)\left(\int g'(x)\,dx\right) \underset{\text{False!}}{=} f(x)g(x). \qquad \text{RFPRF}$$

So, the next time you see the integral $\int x/(x^2+1)\,dx$, avoid

$$\int \frac{x}{x^2+1}\,dx \;=\; \int x\cdot\frac{1}{x^2+1}\,dx \underset{\text{False!}}{=} \frac{1}{2}x^2\arctan(x) + C, \qquad \text{RFPRF}$$

and do it the old-fashioned way, making the substitution $u = x^2+1$:

$$\int \frac{x}{x^2+1}\,dx \;=\; \frac{1}{2}\int u^{-1}\,du \;=\; \frac{1}{2}\ln|u| + C \;=\; \ln\sqrt{x^2+1} + C.$$

NAD = **N**ot an **A**nti**D**erivative of the integrand

Perhaps this error is often committed in haste, out of wishful thinking that, if it were true, would streamline the amount of work it takes to achieve an answer. There are, however, several interesting special cases of this error which I label CADD and RFPRF. Here are some instances of the kinds of mistakes I have labeled NAD:

$$\int \frac{dx}{x^2-4} \underset{\text{False!}}{=} \ln|x^2-4| + C. \qquad \text{NAD}$$

$$\int \frac{dx}{\sqrt{x^2+1}} \underset{\text{False!}}{=} \arctan\sqrt{x^2+1} + C. \qquad \text{NAD}$$

$$\int e^{x^2}\,dx \underset{\text{False!}}{=} \arctan\sqrt{x^2+1} + C. \qquad \text{NAD}$$

This error really is highly preventable for the student who does a good job differentiating. That is because any antiderivative statement, taken in reverse, is a statement about differentiation. If one knows this, and knows to use the chain rule when differentiating $\ln|x^2-4|$, a simple check will reveal

$$\frac{d}{dx}\ln(x^2-4) = \frac{1}{x^2-4}\cdot 2x, \qquad \text{not} \qquad \frac{1}{x^2-4},$$

which was desired. If one is careful to always check a proposed antiderivative by differentiating it, many NAD errors will be avoided.

CADD = **C**onfusing **A**nti**D**erivative and **D**erivative

The most common instance of this involves $\frac{1}{x}$ and $\ln x$. Sometimes students write

$$\int \ln x\,dx \underset{\text{False!}}{=} \frac{1}{x},$$

though there are other (perhaps more common) places to see the error, such as in this integral, where some students fail to see the opportunity at a direct substitution, and instead will integrate by parts:

$$\int x^{-1}\ln x\,dx.$$

Correct accounting while integrating by parts can lead to the correct answer for this integral, but not if one writes

$$\begin{array}{l|l} u = x^{-1} & dv = \ln x\,dx \\ du = -x^{-2}\,dx & v \underset{\text{False!}}{=} x^{-1} \end{array}$$

which (more subtly than above) asserts, again, that x^{-1} is the antiderivative of $\ln x$ when it is actually the derivative.

Other common instances of this mistake:

$$\text{when } dv = e^{-2x}, \text{ and one takes } v = \int dv \underset{\text{False!}}{=} -2e^{-2x} \qquad \text{CADD}$$

$$\text{when } dv = \cos(x), \text{ and one takes } v = \int dv \underset{\text{False!}}{=} -\sin(x) \qquad \text{CADD}$$

WLINV = **W**rong **L**imits of **I**ntegration in the **N**ew **V**ariable

A substitution is appropriate in many integrals, such as $u = x - 1$ in the following:

$$\int \sqrt{x-1}\,dx \;=\; \int \sqrt{u}\,du \;=\; \frac{2}{3}u^{3/2} + C \;=\; \frac{2}{3}(x-1)^{3/2} + C.$$

But one must take care to move the limits of integration, when we have a definite integral.

$$\int_1^2 \sqrt{x-1}\,dx \;=\; \int_0^1 \sqrt{u}\,du \;=\; \frac{2}{3}u^{3/2}\Big|_0^1 \;=\; \frac{2}{3}.$$

Were we to neglect how the change in variable affects the limits of integration, we might write

$$\int_1^2 \sqrt{x-1}\,dx \underset{\text{False!}}{=} \int_1^2 \sqrt{u}\,du \;=\; \frac{2}{3}u^{3/2}\Big|_1^2 \;=\; \frac{2}{3}(2^{3/2}-1) \;\doteq\; 1.21895. \qquad \text{WLINV}$$

Of course, people who concern themselves only with whether or not they get the right answer might point out that, as in the indefinite integral, you can return to the original variable x and achieve this goal:

$$\int_1^2 \sqrt{x-1}\,dx \underset{\text{False!}}{=} \int_1^2 \sqrt{u}\,du \;=\; \frac{2}{3}u^{3/2}\Big|_1^2 \underset{\text{False!}}{=} \frac{2}{3}(x-1)^{3/2}\Big|_1^2 \;=\; \frac{2}{3}. \qquad \text{WLINV}$$

As a general rule, the *end* in mathematics does not *justify the means*. Mathematics is a language used to explain a solution every bit as much as it is a set of tools for finding that solution, and bad writing is, and always will be, bad writing. One should not give an explanation that contains falsehoods, even if, sometimes (as is the case in this instance), two *wrongs* make a *right*. The appropriate way, in instances where we return to the original variable, to do definite integration is to change limits of integration twice:

$$\int_1^2 \sqrt{x-1}\,dx \;=\; \int_0^1 \sqrt{u}\,du \;=\; \frac{2}{3}u^{3/2}\Big|_0^1 \;=\; \frac{2}{3}(x-1)^{3/2}\Big|_1^2 \;=\; \frac{2}{3}.$$

If one wishes to avoid determining, if only for the intermediate steps, the appropriate limits of integration in the new variable, an acceptable alternative might be the following:

$$\int_1^2 \sqrt{x-1}\,dx \;=\; \int_a^b \sqrt{u}\,du \;=\; \frac{2}{3}u^{3/2}\Big|_a^b \;=\; \frac{2}{3}(x-1)^{3/2}\Big|_1^2 \;=\; \frac{2}{3}.$$